1. a) show that $SNR_{\text{voxel}} \propto \Delta x / \sqrt{BW_{\text{voxel}}}$ for constant phase encoding parameters

signal $\propto \Delta x$

$noise \propto \frac{1}{T_s}$

$T_s = N_x \Delta t$

$BW_{\text{read}} = \frac{1}{\Delta t}$

$T_s = \frac{N_x}{BW_{\text{read}}}$

$BW_{\text{voxel}} = \frac{BW_{\text{read}}}{N_x}$

$noise \propto \sqrt{BW_{\text{voxel}}}$

$SNR \propto \frac{\Delta x}{\sqrt{BW_{\text{voxel}}}}$

b) When $\Delta x$ is halved with a corresponding doubling of $G_z$

We know that

$BW_{\text{voxel}} = \frac{BW_{\text{read}}}{N_o}$

$BW_{\text{read}} = \frac{1}{\Delta t}$

$BW_{\text{voxel}} = \frac{1}{\Delta t N_o}$

$FOV = L = N_o \Delta x$

For case 7 have $N_o$ and $\Delta t$ so there is no change in the bandwidth per voxel. For case 8 have $2N_o$ and $\Delta t/2$ so there is again the bandwidth per voxel will stay the same. In both cases, the signal strength from each voxel will be cut in half since $\Delta x$ is halved. Since the signal is halved but the noise which is proportional to the square root of the bandwidth is unchanged this will result in the signal to noise ratio to be cut in half as well. For case 7 the field of view will be halved since $\Delta x$ is halved with no change in $N_o$ but in case 8 there will be no change since $N_o$ is doubled and $\Delta x$ is halved.
c) If $\Delta x$ is halved by increasing $N_s$ without a corresponding doubling of $G_z$ the bandwidth per voxel will be halved. The voxel signal will also be halved. Since the signal is halved but the noise which is proportional to $\sqrt{BW}$ the SNR/voxel will be reduced by a factor of $\sqrt{2}/2 = 1/\sqrt{2}$.

d) One way to double $\Delta x$ is to double the field of view while holding $N_s$ constant and keeping the same gradient and doubling the bandwidth. Another way is to double the field of view while holding $N_s$ constant and halving the gradient.

e) Given:

\begin{align*}
  FOV_{\text{read}} &= 256\text{mm} \\
  N_s &= 256 \\
  G_z &= 2.5mT/m \\
  G_{\text{max}} &= 15mT/m \\
  T_s^* &= 20\text{ms}
\end{align*}

Would like to reduce $\Delta x$ to $\Delta x/2$ while getting optimal SNR/voxel.

\[ \Delta x = \frac{L_x}{N_x} = \frac{256\text{mm}}{256} = 1\text{mm} \]

\[ BW_{\text{read}} = \frac{4G_zL_x}{T_s} = \frac{(42\text{MHz}/T)(2.5mT/m)(256\text{mm})}{20\text{ms}} = 27\text{kHz} \]

\[ \Delta t = \frac{1}{BW_{\text{read}}} = \frac{1}{27\text{kHz}} = 3.7e^{-5}\text{ sec} \]

\[ T_s = N_s\Delta t = 9.4\text{ms} \]

know that the signal to noise ratio for a voxel is proportional to

\[ \text{SNR/voxel} \propto \Delta x\sqrt{T_s} \]

in order to optimize the signal to noise ratio when halving $\Delta x$ want to maximize $T_s^*$. Assuming that $T_s \leq T_s^*$ can optimize $T_s$ by setting it equal to $T_s^*$. So have

\[ T_s = T_s^* = 20\text{ms} \]

\[ \frac{\Delta x}{2} = 0.5\text{mm} \]

Are constrained by $G_{\text{max}}$. So must check to see that G is below $G_{\text{max}}$. 

\[ \Delta t = \frac{1}{BW_{\text{read}}} = \frac{1}{\gamma G_x L_x} \]

\[ G_x = \frac{1}{\Delta t \gamma L_x} \]

\[ \Delta t = \frac{T_r}{N_x} \]

\[ N_x = \frac{2L_x}{\Delta x} \]

since halving \( \Delta t \)
So have

\[ G_x = \frac{1}{\frac{\Delta x}{2} T_r \gamma} \]

\[ G_x = \frac{1}{(0.5 \text{mm})(20 \text{ms})(42 \text{MHz}/T)} = 2.4 mT/m \]

which is below \( G_{\text{max}} \).

2. a) \[ \phi(t) = \int \omega(t) dt = \gamma \int B(t) dt = \gamma \int G_x(t)x(t) dt \]
for constant gradient and stationary spins then
\[ \phi(t) = \gamma G_x x \int dt \]

for lobe A, \( G_x = G \)
\[ \phi_A(t) = \gamma G_x \int_0^t dt \]
\[ \phi_A(t) = \gamma G_x t \]

for lobe B, \( G_x = -2G \)
\[ \phi_B(t) = -2\gamma G_x \int_\tau^{2\tau} dt \]
\[ \phi_B(t) = -2\gamma G_x (2\tau - \tau) \]
\[ \phi_B(t) = -2\gamma G_x \tau \]

for lobe C, \( G_x = G \)
\[ \phi_C(t) = \gamma G_x \int_\tau^{2\tau} dt \]
\[ \phi_C(t) = \gamma G_x (t - 2\tau) \]
so the phase for $2\tau < t < 4\tau$ is

$\phi = \phi_a(t) + \phi_b(t) + \phi_c(t)$

$\phi = \gamma G x - 2\gamma G \tau + \gamma G x (t - 2\tau)$

$\phi = -3\gamma G x + \gamma G x t$

b) $\phi(t) = \int \omega(t) dt = \gamma \int B(t) dt = \gamma \int G_x(t)x(t) dt$

for constant velocity

$x(t) = x_0 + v_x t$

for constant gradient then

$\phi(t) = \gamma G \int x(t) dt = \gamma G \int (x_0 + v_x t) dt$

for lobe A, $G_x = G$

$\phi_A(t) = \gamma G \int_{0}^{t} (x_0 + v_x t) dt$

$\phi_A(t) = \gamma G \left( x_0 t + \frac{1}{2} v_x t^2 \right)$

$\phi_A(t) = \gamma G \left( x_0 \tau + \frac{1}{2} v_x \tau^2 \right)$

for lobe B, $G_x = -2G$

$\phi_B(t) = -2\gamma G \int_{\tau}^{2\tau} (x_0 + v_x t) dt = -2\gamma G \left( x_0 t + \frac{1}{2} v_x t^2 \right)_{\tau}^{2\tau}$

$\phi_B(t) = -2\gamma G \left( x_0 2\tau + \frac{1}{2} v_x (2\tau)^2 \right) - \left( x_0 \tau + \frac{1}{2} v_x \tau^2 \right)$

$\phi_B(t) = -2\gamma G \left( x_0 2\tau + \frac{1}{2} v_x 4\tau \right) - \left( x_0 \tau + \frac{1}{2} v_x \tau^2 \right)$

$\phi_B(t) = -2\gamma G \left( x_0 \tau + \frac{3}{2} v_x (\tau)^2 \right)$

for lobe C, $G_x = G$

$\phi_C(t) = \gamma G \int_{2\tau}^{t} (x_0 + v_x t) dt$

$\phi_C(t) = \gamma G \left( x_0 t + \frac{1}{2} v_x t^2 \right)_{2\tau}^{t}$

$\phi_C(t) = \gamma G \left( x_0 t + \frac{1}{2} v_x t^2 \right) - \left( x_0 2\tau + \frac{1}{2} v_x (2\tau)^2 \right)$
so the phase for $2\pi < t < 4\pi$ is
\[
\phi = \phi_A(t) + \phi_B(t) + \phi_C(t)
\]
\[
\phi = \gamma G \left( x_o \tau + \frac{1}{2} v_x \tau^2 \right) - 2\gamma G \left( x_o \tau + \frac{3}{2} v_x (\tau)^2 \right) + \gamma G \left( x_o t + \frac{1}{2} v_x t^2 \right) - \left( x_o 2\tau + \frac{1}{2} v_x (2\tau)^2 \right)
\]
\[
\phi = \gamma G \left( x_o \tau - 2 x_o \tau - x_o 2\tau + \frac{1}{2} v_x \tau^2 - \frac{6}{2} v_x (\tau)^2 - \frac{1}{2} v_x (2\tau)^2 \right) + \gamma G \left( x_o t + \frac{1}{2} v_x t^2 \right)
\]
\[
\phi = -3\gamma G \left( x_o \tau - \frac{3}{2} v_x (\tau)^2 \right) + \gamma G \left( x_o t + \frac{1}{2} v_x t^2 \right)
\]